

Monday, November 3, Lecture 27: Hilbert Spaces II

Last time...

Hilbert space $\mathcal{H}$ = Vector space with inner product, containing all convergent sums.

A collection of orthonormal vectors $\{u_k\}$ is an orthonormal basis of $\mathcal{H}$ if it satisfies any of these (equivalent) conditions:

① $u^\perp = \{0\}$, $u^\perp := \{v \in \mathcal{H} : \langle v, u_k \rangle = 0 \text{ for all } k\}$

② $f = \sum_k \langle u_k, f \rangle u_k$ for any $f \in \mathcal{H}$

③ $\|f\|^2 = \sum_k |\langle u_k, f \rangle|^2$.

↖ will unpack today!

3 examples:

• $\mathbb{C}^n$ with dot product $\leftarrow \dim = n$

• $\ell^2(\mathbb{Z})$ with "dot product" $\leftarrow \dim = \infty$

$$\langle v, w \rangle = \sum_{k=-\infty}^{\infty} \overline{v(k)} w(k)$$

• $\delta_j \in \ell^2(\mathbb{Z})$: $\delta_j(k) = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k \end{cases}$

• $\{\delta_k : k \geq 0 \text{ and } k \in \mathbb{Z}\}$ is not an orb, since $\delta_{-1} \in u^\perp$

• $\{\delta_k : k \in \mathbb{Z}\}$ is an orb. Take any $f \in \ell^2(\mathbb{Z})$. Then

$$f = \sum_k f(k) \delta_k$$

$$(\dots, f(-1), f(0), f(1), \dots) = \dots \delta_{-1} f(-1) + \delta_0 f(0) + \delta_1 f(1) + \dots$$

• $L^2([-π, π])$ with inner product

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{f(x)} g(x) dx.$$

• $\{\sqrt{2} \sin kx : k \in \mathbb{N}\}$ is not an onb, since $f(x) = x^2$ has $f \in U^\perp$.

• $\{e^{ikx} : k \in \mathbb{Z}\}$ is an onb

• ② in onb definition gives the exponential Fourier series: if $f \in L^2([-π, π])$,

$$f = \sum_{k=-\infty}^{\infty} \langle e^{ikx}, f \rangle e^{ikx}$$

Notice:
 $f \in L^2(\mathbb{Z})$

→ We often think of these Fourier coefficients as

$$\begin{aligned} \hat{f}(k) &:= \langle e^{ikx}, f \rangle \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikx} f(x) dx \end{aligned}$$

• ③ is often called Parseval's Theorem*

$$\|f\|^2 = \sum_{k=-\infty}^{\infty} |\langle e^{ikx}, f \rangle|^2 = \sum_{k=-\infty}^{\infty} |\hat{f}(k)|^2.$$

"The infinite-dimensional Pythagorean Theorem"

*More interesting!

• $\{1\} \cup \{\sqrt{2} \sin kx : k \in \mathbb{N}\} \cup \{\sqrt{2} \cos kx : k \in \mathbb{N}\}$ is an orthonormal basis.

[Note: there are many conventions for this, usually stemming from different inner products.

Compare: $\langle \cdot, \cdot \rangle$ vs $\langle \cdot, \cdot \rangle_{\text{new}}$

$$\langle 1, 1 \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} 1^2 dx = 1 \quad \text{normalized} \checkmark$$

$$\langle 1, 1 \rangle_{\text{new}} = \int_{-\pi}^{\pi} 1^2 dx = 2\pi \quad \text{not normalized!}$$

but $\frac{1}{\sqrt{2\pi}}$ is normalized.

[Just check the normalization.

• It does not change much to rescale the inner product on $L^2([-\pi, \pi])$. Pick

$$\langle f, g \rangle_{\Delta} := \frac{1}{\pi} \int_{-\pi}^{\pi} \overline{f(x)} g(x) dx$$

• $\Rightarrow \left\{ \frac{1}{\sqrt{2}} \right\} \cup \{ \cos kx : k \in \mathbb{N} \} \cup \{ \sin kx : k \in \mathbb{N} \}$ is an orthonormal basis.

• ② gives the trigonometric Fourier Series.

$$f = \left\langle \frac{1}{\sqrt{2}}, f \right\rangle_{\Delta} + \sum_{k=1}^{\infty} \langle \cos kx, f \rangle_{\Delta} \cos kx + \langle \sin kx, f \rangle_{\Delta} \sin kx$$

$$= \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$$

• ③ is Parseval's Theorem:

$$\begin{aligned}\|f\|_{\Delta}^2 &= \left| \langle \frac{1}{\sqrt{2}}, f \rangle_{\Delta} \right|^2 + \sum_{k=1}^{\infty} \left| \langle \cos kx, f \rangle_{\Delta} \right|^2 + \left| \langle \sin kx, f \rangle_{\Delta} \right|^2 \\ &= \left| \frac{a_0}{2} \right|^2 + \sum_{k=1}^{\infty} |a_k|^2 + |b_k|^2\end{aligned}$$

* A subtlety ... what do we actually mean when we write
$$f = \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ikx} \quad ? \quad (\text{or any other way})$$

• If $f \in L^2([-\pi, \pi])$ then the Fourier partial sums

$$S_N(x) = \sum_{k=-N}^N \hat{f}(k) e^{ikx} = \sum_{k=-N}^N \langle f, e^{ikx} \rangle e^{ikx}$$

converges in L^2 to f , meaning

$$\lim_{N \rightarrow \infty} \|f(x) - S_N(x)\| = 0.$$

← aka mean-square
convergence, convergence
in energy, ...

• So: they are equal as $L^2([-\pi, \pi])$ functions.

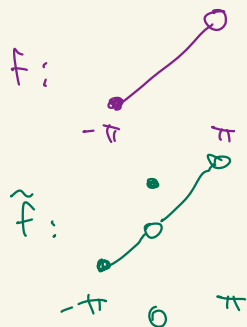
This can be different from pointwise convergence,
where $f = \tilde{f}$ pointwise if

$$f(x) = \tilde{f}(x) \text{ for all } x \in [-\pi, \pi]$$

Ex: $f, \tilde{f} \in L^2([-\pi, \pi])$.

$$f(x) = x,$$

$$\tilde{f}(x) = \begin{cases} x & x \neq 0 \\ 1 & x = 0 \end{cases}$$



Different functions, but the same in L^2 :

$$\|f - \tilde{f}\|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - \tilde{f}(x)|^2 dx = \frac{1}{2\pi} \int_0^0 1 dx = 0$$

$f - \tilde{f}$ is a function that is 0 almost everywhere, except for a single point where it is 1. This is impossible to integrate!

In an Hilbert space, $u=v$ if and only if $\|u-v\|=0$.

$\Rightarrow f = \tilde{f}$ as L^2 functions

This subtlety is still present when we consider Fourier series. (Later: Gibbs phenomenon)

There is another way to think about Fourier Series.

Recall: adjoint. If $A: \mathcal{X}_1 \rightarrow \mathcal{X}_2$ is a linear map,

then $A^*: \mathcal{X}_2 \rightarrow \mathcal{X}_1$ is the linear map

$$\langle f, Ag \rangle = \langle A^* f, g \rangle \quad \text{for all } g \in \mathcal{X}_1, f \in \mathcal{X}_2.$$

- A unitary map is one s.t.

$$U^* U = \mathbb{1} \quad \text{the identity operator}$$

$$U U^* = \mathbb{1}$$

Fundamental property: if $f, g \in \mathcal{H}$, U

$$\langle Uf, Ug \rangle = \langle f, U^* U g \rangle = \langle f, g \rangle$$

aka U is
an isometry
iso = same
metry = measure

Special case:

$$(*) \quad \|Uf\|^2 = \langle Uf, Uf \rangle = \langle f, f \rangle = \|f\|^2$$

Now: let $U: L^2([-\pi, \pi]) \rightarrow \ell^2(\mathbb{Z})$

$$f \mapsto \hat{f}$$

then $(*)$ is

$$\|f\|^2 = \|Uf\|^2 = \|\hat{f}\|^2$$

Parseval's Thm II:

Let $f, g \in L^2([-\pi, \pi])$. Then the map $U(f) = \hat{f}$ is unitary:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{f(x)} g(x) dx = \sum_{k=-\infty}^{\infty} \overline{\hat{f}(k)} \hat{g}(k)$$

$$\langle f, g \rangle_{L^2([-\pi, \pi])} = \langle \hat{f}, \hat{g} \rangle_{\ell^2(\mathbb{Z})}$$

Unitaries are special change of bases...
so what operators are we diagonalizing?
Derivative operators: